JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 21, No. 4, July–August 1998

Using Bifurcation Methods to Aid Nonlinear Dynamic Inversion Control Law Design

D. M. Littleboy* and P. R. Smith†
Defence Evaluation and Research Agency, Bedford, England MK41 6AE, United Kingdom

The use of bifurcation analysis as an aid to nonlinear control law design and analysis is demonstrated by application to a hypothetical, high-angle-of-attack combat aircraft. Bifurcation analysis determines the equilibrium flight conditions of the unaugmented aircraft as a function of the elevator with all lateral controls set at zero. These results are used to identify instabilities that cause aircraft departure into spin regions. The removal of these departures, and hence the onset of spins, is achieved using a nonlinear dynamic inversion-based control law until the lateral control surfaces saturate, thus extending the stable, decoupled flight envelope. The work illustrates how bifurcation analysis may be used to formulate control law specifications for an unaugmented aircraft and to validate the control law by analyzing the resulting augmented aircraft model. The results also indicate the global stabilization properties of nonlinear dynamic inversion.

Nomenclature

b	= wing span
C_l	= nondimensional rolling moment coefficient
C_m	= nondimensional pitching moment coefficient
C_n	= nondimensional yawing moment coefficient
C_y	= nondimensional side force coefficient
C_z	= nondimensional normal force coefficient
c	= reference chord length
H	= height
I_{xx}, I_{yy}, I_{zz}	= moments of inertia
I_{xz}	= product of inertia
L, M, N	= rolling, pitching, and yawing moments
$L_{\bar{\eta}}$	= rolling moment due to differential elevator
L_{ξ}	= rolling moment due to aileron
m	= mass
p, q, r	= body axes roll, pitch, and yaw rates
$\bar{q}$	= freestream dynamic pressure
S	= reference wing area
u	= control parameter
u, v, w	= velocity relative to the air along x, y, z axes
V	= total velocity
X, Y	= aircraft position in space in X and Y axes directions
x	= state vector
α	= angle of attack
β	= sideslip angle
η	= elevator deflection
$\bar{\eta}$	= differential elevator deflection
ξ	= aileron deflection
ρ	= air density
ς	= rudder deflection
ϕ, θ, φ	= roll, pitch, and yaw angles

Subscript

d	= demanded quantity

Received July 30, 1997; presented as Paper 97-3717 at the AIAA Atmospheric Flight Mechanics Conference, New Orleans, LA, Aug. 11–13, 1997; revision received Feb. 16, 1998; accepted for publication Feb. 20, 1998.

*Research Scientist, Flight Management and Control Department. Member AIAA.

†Principal Scientist, Flight Management and Control Department. Member AIAA.

Introduction

MODERN fighter aircraft are capable of performing rapid maneuvers and of flying at high-incidence angles where the dynamic behavior is highly nonlinear; these nonlinearities are due to inertial or aerodynamic effects. This nonlinear dynamic behavior makes the aircraft susceptible to departures and divergences in longitudinal and lateral motions.

Linear analysis methods give only local information and are, thus, limited in these flight regions. Certain phenomena exist only in nonlinear systems, such as limit cycle effects and multiple solutions occurring per degree of freedom. Therefore, effective global nonlinear analysis tools are required to enable the dynamic behavior of an aircraft to be analyzed throughout the whole flight envelope.

Bifurcation analysis techniques have been widely used to investigate the nonlinear dynamical behavior of aircraft[1–16] since the first high-angle-of-attack study by Mehra et al.[1] Since then, bifurcation methods have been used to analyze the complex nonlinear phenomena for mathematical models of the F-4 (Refs. 2–4), F-14 (Refs. 5 and 6), F-15 (Refs. 7–9), F-16 (Refs. 10 and 11), and F-18 (Ref. 12) fighters and of the German–French Alpha-jet.[13]

The majority of these studies have focused on the analysis of the unaugmented aircraft, particularly the prediction of nonlinear phenomena such as wing rock, autorotation, deep stall, and spinning behavior. These predictions are usually validated using time simulation. In some studies, the open-loop bifurcation results have been used to determine spin recovery techniques[2,5,13] using conventional controls, whereas spin recovery using pitch thrust vectoring control has also been considered.[8]

Only two studies[7,10] have analyzed the effects of representative control augmentation systems using bifurcation methods. These studies demonstrate the use of bifurcation methods for validating the control law design criteria such as the effectiveness of stabilization, command following, and angle-of-attack limiters. It is, therefore, apparent that bifurcation analysis is a powerful tool for analyzing the highly nonlinear characteristics of both unaugmented aircraft models and those with control augmentation systems.

Less work has been performed in using these bifurcation results to either design control laws or to modify existing control systems. The design of simple aileron–rudder interconnection schemes has been developed[1,2,16] to avoid unwanted dynamic characteristics or to enhance lateral maneuvers. Jahnke and Culick[5] demonstrated the use of roll rate and sideslip feedback to the ailerons to remove wing rock and suggest the use of control deflection limiters to prevent the aircraft from entering into known spin regions. Similarly, Planeaux et al.[7] were able to delay the onset of wing rock using roll rate feedback to the aileron or differential tail.

In more recent works, Liebst and DeWitt[9] investigated numerous combinations of roll rate, angle of attack, and sideslip feedback to stabilator, aileron, and rudder to delay the onset of wing rock. These feedback strategies are based on detailed sensitivity analysis to determine which stability derivatives are dominant in wing rock. Wang et al.[11] discretized the nonlinear aircraft equations of motion using a fourth-order Runge–Kutta scheme to enable pole placement to achieve desired levels of response frequency and damping. Sensitivity analysis was again used to develop simple, linear control laws to feedback angle of attack and pitch rate to elevator, sideslip and roll rate to aileron, and sideslip and yaw rate to rudder. Both methods[9,11] relied on detailed stability and sensitivity analysis for the design of the control system for a single criterion.

This study uses bifurcation analysis in the iterative design of a nonlinear, full envelope control law to both stabilize the aircraft and to remove the undesired nonlinear effects. This extends the use of bifurcation methods from open-loop analysis to closed-loop control law design and validation. It is not the aim of this paper to again demonstrate that bifurcation analysis is an efficient method for analyzing nonlinear aircraft dynamic behavior. Nor is it the aim to design simple, linear control laws to satisfy a single design criterion on a specific region of the flight envelope being analyzed.

To achieve these design goals, a nonlinear dynamic inversion- (NDI-) based control law method is used. Much theoretical work has been performed on the development of NDI methods and their application to aircraft control problems.[17–22] However, the stability properties of NDI have not yet been proven. The results in this paper of the bifurcation analysis of an NDI control law over the full range of control input indicate that NDI does, indeed, provide global stability with respect to a slowly varying control parameter, but not in the sense of robustness to external disturbances, such as gusts, or to large, abrupt control inputs.

First, an outline of bifurcation theory is given. The unaugmented aircraft model used for this work is then described, detailing its nonlinear characteristics. This unaugmented aircraft model is analyzed using bifurcation methods, and these open-loop bifurcation results are used to define control law design specifications of global stabilization to remove the onset of spinning behavior. The NDI control law design method is described and is then used to design a control law to meet these design specifications. It is shown how the augmented aircraft model remains in a trimmed, straight, and level flight condition for a greatly extended flight envelope until the lateral controls saturate.

Bifurcation Analysis

The dynamic characteristics of complex systems can be investigated through the solution of their governing nonlinear differential equations. This analysis is performed by calculating all of the steady states (also known as equilibrium solutions) of the differential equations. The stability of the dynamic system at each steady-state condition is determined by the local stability of the linearized system around the steady state. A system is defined to be locally stable if the real part of all of the linearized eigenvalues are negative. Conversely, the system is locally unstable if the real part of any of the linearized eigenvalues is positive.

The steady states are continuous functions of the parameters of the system, provided the local linearized system is regular, i.e., nonsingular. Stability changes are characterized by a crossing of the imaginary axis of one or more eigenvalues between successive steady states. These stability changes result in qualitative changes in the system dynamics and are called bifurcations. There are many types of bifurcation phenomena, which result in different dynamic responses of the system. A particularly important example is the Hopf bifurcation, which occurs when a complex-conjugate pair of eigenvalues crosses the imaginary axis. At the exact point of crossing, the complex eigenvalue has a zero real part, implying pure oscillatory behavior. Therefore, a Hopf bifurcation indicates a point at which limit cycle behavior starts. This periodic motion needs to be investigated by time integration and is an extension to the basic bifurcation analysis of steady states.

Thus, for the analysis of nonlinear aircraft models, the steady-state solutions are calculated as continuous functions of control surface deflections. The steady states are usually calculated for the full authority range of a single control surface, while all other control surfaces are held constant, resulting in a global picture of the steady-state behavior of the aircraft. Bifurcation points indicate a change in the aircraft dynamics, generally called a departure. Example departures are jumps from a controlled stable region to an unstable spin or to wing rock behavior. Whereas bifurcation analysis results in a global steady-state representation of the aircraft, it gives no information about the transient motions. Therefore, the exact nature of the dynamic response of the aircraft following a bifurcation needs to be investigated through time simulation.

To calculate the all of the steady states and bifurcation points of a system, numerical continuation methods are used. Consider the general nonlinear system of equations $\dot{x} = F(x, u)$, where x is the state vector and u is a scalar parameter (such as one of the aircraft control surfaces). To find the steady-state solutions, the algebraic equation $F(x, u) = 0$ is solved using gradient-based methods as the parameter u varies. The continuation to a successive equilibrium solution is achieved by first calculating a unit tangent vector at the current solution. The increments of the state vector and the parameter along the trajectory are defined by projections of the unit tangent vector. The next equilibrium point along the trajectory is calculated by solving an equation in which the trajectory is parameterized by the curve length to avoid singularity problems. Corrections are then made to the solution until it satisfies a prescribed accuracy condition. This continuation method is based on that used in an alternative analysis package.[23]

Aerodynamic Model

There is some freedom as to the fidelity of the aircraft model that is chosen for the bifurcation analysis, provided the quantification of the results is based on any simplifications or assumptions made. For simulation purposes, a full six-degree-of-freedom aerodynamic model would be governed by 12 equations of motion for the states $x = [\alpha, \beta, p, q, r, V, \theta, \phi, \varphi, H, X, Y]^T$, with a full aerodynamic database including low-angle-of-attack effects, rotary balance data, static tests data, etc. Such comprehensive aircraft models can be very complex and may be computationally intensive to analyze using numerical bifurcation methods.

However, aircraft models of high order have been successfully analyzed,[5–13] often with some simplifications made due to the nature of the investigations. The 12th-order equations can be simplified to eighth order by removing the three equations for aircraft position in space and the heading equation, the remaining states being $x = [\alpha, \beta, p, q, r, V, \theta, \phi]^T$. The eighth-order set of equations are those usually used for the bifurcation analysis of nonlinear aircraft characteristics. Further simplifications can be made to the eighth-order equations when spatial maneuvers with high rates of rotation are considered. For these considerations, it is assumed that velocity is constant and that gravitational effects are negligible, thus, reducing the aircraft model to five equations of motion for $x = [\alpha, \beta, p, q, r]^T$. It has been shown[6] that the fifth- and eighth-order aircraft models predict very similar occurrences of roll coupling, but that the steady-state rotation rates in a spin can be considerably overestimated.

Aerodynamic model simplifications can also be made, such as by using smoothing methods[6] on the aerodynamic data or by omitting oscillatory and rotary balance data. Again, care is required for the interpretation of the results because rotary balance data are needed for the accurate modeling of the aerodynamics in a spin.[6] It is apparent that whatever order or aerodynamic fidelity of aircraft model is used, time simulation is inevitably required to validate theoretical results because bifurcation results give no information on the transient behavior of the aircraft.

The aerodynamic model used for these studies is called the hypothetical high-incidence research model (HHIRM) and has a number of features that lend themselves to analysis by bifurcation methods. The aerodynamics are constructed using a generic set of nonlinear spline function blocks, which are able to represent highly complicated nonlinear aircraft characteristics, such as loss of aerodynamic stability due to shadowing of vertical tail, loss of lateral aerodynamic stability due to vortex breakdown, autorotation in roll and yaw, and aerodynamic asymmetry in roll and yaw.

The aerodynamic conditions present in high-angle-of-attack stall and spin conditions are modeled on those that would be obtained from static, forced oscillations and rotary balance tests. The aircraft model has strong aerodynamic coupling between the longitudinal and lateral motion modes due to the dependence of rotary derivatives on elevator deflection and of moment asymmetry on angle of attack.

The aircraft model has been constructed for an F-16 style configuration. The control surfaces modeled are symmetric elevator, differential elevator, ailerons, and rudder. Figure 1 shows dependencies of the aerodynamic coefficients on angle of attack. The lateral coefficients are plotted for different sideslip angles in the range $-15 \leq \beta \leq 15$ deg, and the longitudinal coefficients are plotted for different elevator angles in the range $-30 \leq \eta \leq 10$ deg. It can be seen that highly nonlinear behavior exists in the high-incidence region. An additional benefit of this aircraft model is that the aerodynamics are smooth, therefore, removing the need to process the aerodynamic data and improving the speed and efficiency of the analysis. Nonsmooth data can present problems to numerical continuation schemes, but curvatures introduced by smoothing techniques can alter the bifurcation results.

The fifth-order set of equations are used to model the aircraft dynamics. The justification for using this reduced-order model is that it is not the aim here to perform full analysis of an unaugmented aircraft to predict the nature of highly nonlinear characteristics. Rather, bifurcation methods are used as an aid to control law design, for which the fifth-order aircraft model is adequate. It also reduces the computation time for the analysis. The equations used are

$$\dot{\alpha} = q - \tan\beta(r\sin\alpha + p\cos\alpha) + \frac{\rho V S C_z \cos\alpha}{2m\cos\beta}$$

$$\dot{\beta} = (p\sin\alpha - r\cos\alpha) + \frac{\rho V S[C_y\cos\beta - C_z\sin\beta\sin\alpha]}{2m}$$

$$\dot{p} = \left\{\left[-\left(\frac{I_{zz}-I_{yy}}{I_{xx}} + \frac{I_{xz}^2}{I_{xx}I_{zz}}\right)qr + \left(1 - \frac{I_{yy}-I_{xx}}{I_{zz}}\right)\frac{I_{xz}}{I_{xx}}pq + \frac{\bar{q}Sb}{I_{xx}}\left(C_l + \frac{I_{xz}}{I_{zz}}C_n\right)\right]\Bigg/\left(1 - \frac{I_{xz}^2}{I_{xx}I_{zz}}\right)\right\} \tag{1}$$

$$\dot{q} = \frac{\bar{q}Sc}{I_{yy}}C_m + \frac{I_{zz}-I_{xx}}{I_{yy}}pr + \frac{I_{xz}}{I_{yy}}(p^2 - r^2)$$

$$\dot{r} = \left\{\left[\left(\frac{I_{xz}^2}{I_{xx}I_{zz}} - \frac{I_{yy}-I_{xx}}{I_{zz}}\right)pq + \left(1 + \frac{I_{zz}-I_{yy}}{I_{xx}}\right)\frac{I_{xz}}{I_{zz}}qr + \frac{\bar{q}Sb}{I_{zz}}\left(\frac{I_{xz}}{I_{xx}}C_l + C_n\right)\right]\Bigg/\left(1 - \frac{I_{xz}^2}{I_{xx}I_{zz}}\right)\right\}$$

which represent the fast modes of the system, namely, the longitudinal short-period mode and the lateral Dutch roll and roll subsidence modes. Note that for this aircraft model it is assumed that thrust is equivalent to drag. The operating condition is set at a constant velocity of 150 m/s and a constant altitude of 5000 m.

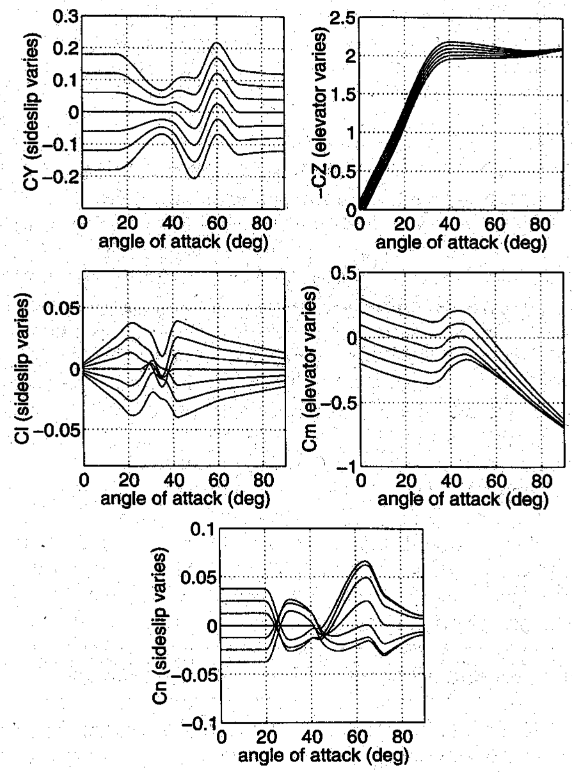

Fig. 1 Aerodynamic coefficients.

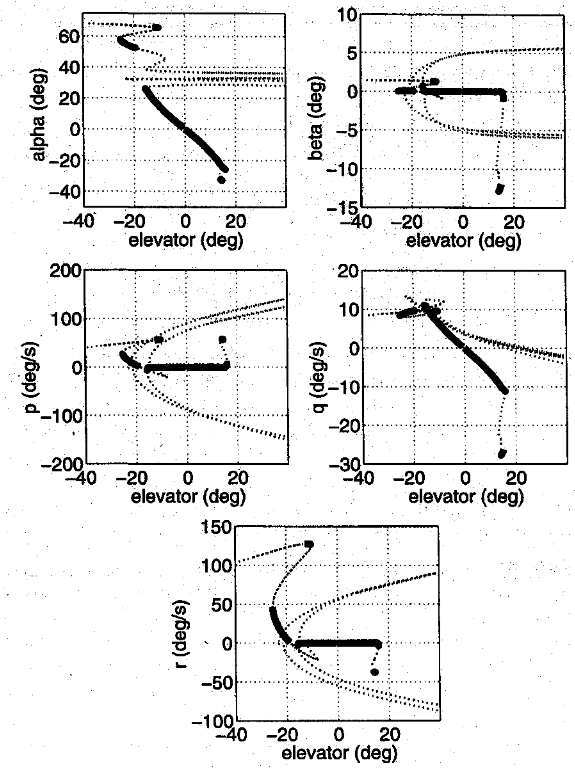

Fig. 2 Steady states for unaugmented aircraft model: ▬, stable, and ----, unstable.

Unaugmented Aircraft Model Results

The unaugmented HHIRM has been analyzed by calculating all of the steady states as the elevator is varied within its authority limits, with all lateral controls set to zero. Figure 2 shows these steady states with elevator deflection plotted against each of the aircraft equation states: angle of attack α, sideslip angle β, roll rate p, pitch rate q, and yaw rate r. It should be noted that the pitch rate solutions are not representative of full six-degree-of-freedom results due to the fixed speed constraint and the omission of gravitational effects.

It is clear from Fig. 2 that the HHIRM has stable, decoupled longitudinal and lateral dynamics in the range $-26 \leq \alpha \leq 26$ deg for $-16 \leq \eta \leq 16$ deg because the values for sideslip angle, roll rate, and yaw rate are all zero. The results are almost linear, as expected at low-incidence regions by inspection of the aerodynamic coefficients in Fig. 1. In Fig. 1, for incidence angle up to $\alpha = 26$ deg with zero sideslip angle, all lateral coefficients are zero. Beyond $\alpha = 26$ deg, nonlinear coupling effects appear in the rolling moment coefficient for zero sideslip and in the side force and yawing moment coefficients for nonzero sideslip.

Above this incidence angle, the steady states become unstable due to a single real eigenvalue becoming positive. The dynamics then represent spinning behavior indicated by a constant angle of attack for large changes of elevator deflection with maximum lateral characteristics of 5-deg sideslip angle, −150-deg/s roll rate, and

−75-deg/s yaw rate. In the range $26 \le \alpha \le 38$ deg, there are three spin regions, all displaying large lateral rotation rates and nonzero sideslip angle for constant incidences. The aerodynamic asymmetry in the aircraft model can be seen due to the asymmetric nature of the spin branches.

In the higher angle-of-attack range $38 \le \alpha \le 70$ deg, the aircraft model is mostly unstable, with small stable regions at $52 \le \alpha \le 58$ and $65 \le \alpha \le 66$ deg. At the limits of these regions, there are Hopf bifurcations at $\alpha = 52$ and $\alpha = 66$ deg indicating the onset of periodic motion in those areas. In the negative angle-of-attack region for $\alpha < -26$ deg, the aircraft again becomes unstable and lateral coupling appears, although with lower roll and yaw rates. For a comprehensive analysis of the aircraft model using these results, time simulation needs to be performed to determine the exact nature of departures, spinning behavior, and periodic motion. Particularly, transient behavior needs to be investigated to determine postdeparture response.

Time simulation is not presented because it is not necessary to characterize exactly the dynamic behavior for the hypothetical aircraft inasmuch as many studies have used bifurcation analysis on real aircraft models. Here, the undesirable nonlinear coupled effects seen in the unaugmented aircraft model results are used as specifications for the design of a full envelope, nonlinear control law.

Note also that it is known that there is a branch of solutions that corresponds to roll coupling that is disconnected from the main trim branch. This has not been displayed because the control law is designed to prevent departures from a steady pitchup maneuver into unstable regions, which does not guarantee the prevention of departure due to external disturbances such as large gusts. Therefore, the aim is to extend the stable, decoupled trim region beyond $\alpha = 26$ deg to remove the departure of the aircraft into the spin regions.

At this stage, the results of bifurcation analysis can be used to focus control law design on the regions most prone to departures from normal controlled flight for any aircraft under consideration. It would be possible to concentrate the control law effort around the departure region using any linear method, but here a nonlinear control law is designed for the whole region being considered.

NDI Control

To prevent departures of the aircraft model into unstable regions, an NDI-based control law is designed. The reason for the use of NDI is that previous control law designs, with bifurcation analysis as a design aid, have used linear methods to concentrate on restricted areas of the flight envelope. This can involve considerable analysis of the aircraft dynamics in that area to determine the exact nature of the behavior and to determine which derivatives or states are producing the instability. Using NDI, a simple, generic nonlinear control law can be designed for the whole region without detailed analysis.

There has been a significant amount of work published on the development of NDI and its application to flight control law design.[17–22] NDI has been shown to provide better results in comparison to standard, gain-scheduled linear control law design methods, particularly for poststall maneuvers. The robust stability and performance of any nonlinear system is difficult to check, and it is suggested[18] that the robustness of NDI may be checked using linearizations of the closed-loop nonlinear system along some sample trajectory. This approach was used[20] to compare the robustness of NDI to an H_∞/μ-synthesis formulation. The work presented here illustrates how bifurcation methods may be used to evaluate the stability robustness of NDI.

The NDI method described here is a simple approach, based on algebraic manipulation of the standard aircraft equations of motion. In theory, NDI gives a response that exactly matches the desired response. The method is developed with respect to standard, textbook rotational and translational equations. Therefore, the aircraft states used here may differ from those considered in the bifurcation analysis because this is a generic derivation of NDI. Also only the three primary control surfaces, elevator, ailerons, and rudder, are considered.

Consider the standard equations that define the rotational accelerations of a conventional symmetric aircraft with one principal control surface for each axis,

$$
\begin{aligned}
I_{xx}\dot{p} &= (I_{yy} - I_{zz})qr + I_{xz}(\dot{r} + pq) + L \\
I_{yy}\dot{q} &= (I_{zz} - I_{xx})rp + I_{xz}(r^2 - p^2) + M \\
I_{zz}\dot{r} &= (I_{xx} - I_{yy})pq + I_{xz}(\dot{p} - qr) + N
\end{aligned} \tag{2}
$$

That is, the rotational accelerations are functions of the aircraft aerodynamics, inertias, body rates, and body accelerations. The nonlinear aerodynamic equations that give rise to the moments, L, M, and N, are a function of the state variables and aerodynamic derivatives and the control deflections and control derivatives and are given by

$$
\begin{aligned}
L &= L_v v + L_p p + L_r r + L_\xi \xi + L_\varsigma \varsigma \\
M &= M_u u + M_w w + M_{\dot{w}} \dot{w} + M_q q + M_\eta \eta \\
N &= N_v v + N_p p + N_r r + N_\xi \xi + N_\varsigma \varsigma
\end{aligned} \tag{3}
$$

Equations (2) and (3) can be manipulated to provide directly decoupled control of the rotational accelerations. First, define

$$
\begin{aligned}
\Gamma &= L_v v + L_p p + L_r r \\
\Lambda &= M_u u + M_w w + M_{\dot{w}} \dot{w} + M_q q \\
\Omega &= N_v v + N_p p + N_r r
\end{aligned} \tag{4}
$$

Equation (3) can then be re-expressed as

$$
\begin{aligned}
L = \Gamma + L_\xi \xi + L_\varsigma \varsigma, \qquad & M = \Lambda + M_\eta \eta \\
N = \Omega + N_\xi \xi + N_\varsigma \varsigma &
\end{aligned} \tag{5}
$$

Equation (5) is substituted into Eq. (2) to give

$$
\begin{aligned}
I_{xx}\dot{p} &= (I_{yy} - I_{zz})qr + I_{xz}(\dot{r} + pq) + \Gamma + L_\xi \varsigma + L_\varsigma \varsigma \\
I_{yy}\dot{q} &= (I_{zz} - I_{xx})rp + I_{xz}(r^2 - p^2) + \Lambda + M_\eta \eta \\
I_{zz}\dot{r} &= (I_{xx} - I_{yy})pq + I_{xz}(\dot{p} - qr) + \Omega + N_\xi \varsigma + N_\varsigma \varsigma
\end{aligned} \tag{6}
$$

Equation (6) is now used to develop the control of the body axis rates. If there is a sensor for $\dot{q}$ and η, indicated by a subscript s, then from Eq. (6)

$$
I_{yy}\dot{q}_s = (I_{zz} - I_{xx})rp + I_{xz}(r^2 - p^2) + \Lambda + M_\eta \eta_s \tag{7}
$$

Similarly, introduce a demand d, then

$$
I_{yy}\dot{q}_d = (I_{zz} - I_{xx})rp + I_{xz}(r^2 - p^2) + \Lambda + M_\eta \eta_d \tag{8}
$$

Subtracting Eq. (7) from Eq. (8) gives

$$
\eta_d = (\dot{q}_d - \dot{q}_s)(I_{yy}/M_\eta) + \eta_s \tag{9}
$$

Thus, provided rotational acceleration and control surface position can be measured, a demanded pitch acceleration can be met, provided pitch inertia and control surface effectiveness are known. The roll and yaw equations in Eq. (6) can be manipulated in a similar manner to that for the pitch axis, except that three assumptions need to be made.

1) The cross product of the inertia term is small.

2) Rolling moment due to the rudder, L_ς, is small in comparison to the rolling moment due to the aileron.

3) Yawing moment due to the aileron, N_ξ, is small in comparison to the yawing moment due to the rudder.

Resulting equations similar to that for pitch [Eq. (9)] are then

$$
\begin{aligned}
\xi_d &= (\dot{p}_d - \dot{p}_s)(I_{xx}/L_\xi) + \xi_s \\
\varsigma_d &= (\dot{r}_d - \dot{r}_s)(I_{zz}/N_\varsigma) + \varsigma_s
\end{aligned} \tag{10}
$$

The expressions in Eqs. (9) and (10) are implemented using a simple first-order response for the body rate accelerations. For the pitch axis, this is

$$\dot{q}_d = (q_d - q_s)q_{\text{bw}} \tag{11}$$

where q_{bw} is the bandwidth for the response. Thus, the commanded pitch rate can be directly mapped into a commanded pitch acceleration. The same approach is applicable for the roll and yaw rate commands. Thus, body rate demands are converted to accelerations, which in turn give the control deflections needed to obtain these accelerations. This is, therefore, a simple method for achieving exact response requirements.

Longitudinal Controller

The results obtained in Fig. 1 for the unaugmented aircraft model showed a departure into spin regimes and much static instability at the extremes of the incidence range. The control law is designed iteratively, beginning with an initial longitudinal controller in the pitch axis only. A schematic of the longitudinal controller is shown in Fig. 3.

The input to the control system is a longitudinal stick, which generates a pitch rate demand to achieve a commanded angle of attack. The NDI method calculates the elevator deflection required to achieve this command using the formulas in Eqs. (9) and (11). The steady states of the augmented aircraft model are calculated as a function of varying longitudinal stick input, but the bifurcation diagrams are again plotted with the elevator on the x axis. This enables a simple comparison between the bifurcation results of the unaugmented and the augmented aircraft models and is valid because the stick input variation effects an elevator deflection. An additional diagram is included to show this stick shaping relationship between the longitudinal stick input and the effected elevator deflection.

The steady-state solutions for the augmented aircraft model with the lateral control surfaces again set to zero are in Fig. 4. Comparing Figs. 2 and 4, it is apparent that the values of the steady-state solutions have not been altered by the longitudinal controller. The main effect has been to improve the stability of the aircraft model in the higher angle-of-attack regions and for the lower negative angle-of-attack region.

That the steady-state equilibria are unaltered is an important result when considering piloted simulation investigation of spin characteristics using bifurcation results for modern fighter aircraft that are statically unstable in pitch. It is difficult for a pilot to perform a pitchup maneuver in a controlled manner to the high-angle-of-attack spin regions for airframes that are statically unstable in pitch. The implementation of a complex control law may introduce additional dynamics to the system, thus, altering the bifurcation plot. The use of a simple NDI control law, which does not introduce additional dynamics and retains the open-loop steady states, allows a pilot to reach spin regimes in a controlled manner, upon which the NDI control law can be switched off, and the spin investigated. More detail can be found in Ref. 24.

The longitudinal controller has not satisfied the original design specifications of stabilizing the region to prevent departures to unstable, coupled spin regimes. To do this the lateral controls must be used to counteract the lateral aerodynamic forces and moments produced at the higher angles of attack.

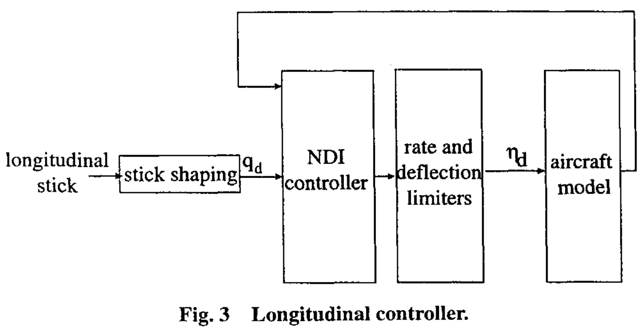

Fig. 3 Longitudinal controller.

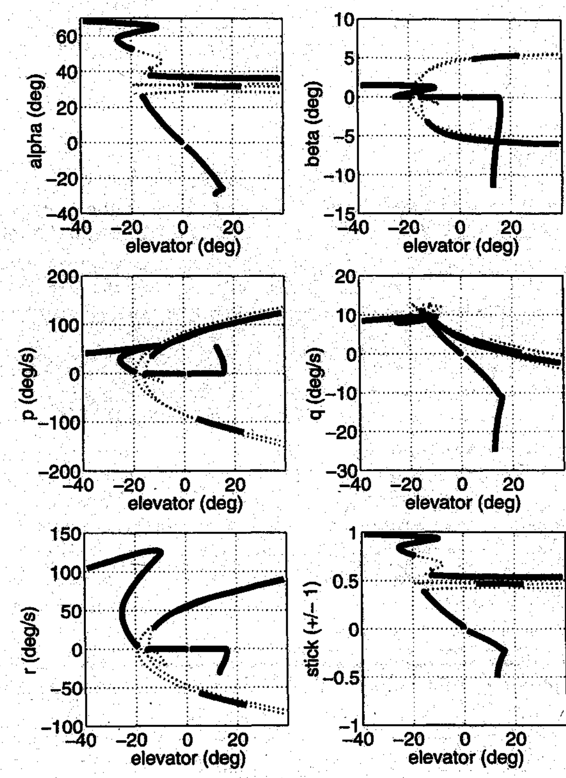

Fig. 4 Steady states for augmented aircraft model (longitudinal controller): ▬, stable, and - - - -, unstable.

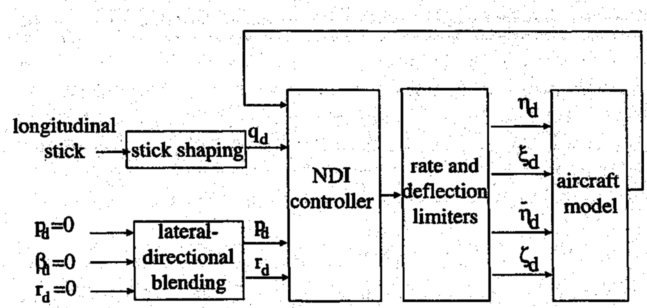

Fig. 5 Longitudinal and lateral controller.

Longitudinal and Lateral Controller

To effect the lateral control surfaces to remove the longitudinal/lateral coupling for high-incidence angle a lateral NDI controller is implemented, with the longitudinal controller remaining the same. A schematic of the longitudinal and lateral controller is shown in Fig. 5. The NDI controller block contains individual NDI control blocks for the roll, pitch, and yaw axes. The inputs to the roll and yaw axes controllers are zero demands for sideslip angle, roll rate, and yaw rate. The yaw axis NDI block generates a rudder demand, but the roll axis NDI block generates both aileron and differential elevator demands. These demands are calculated using a ratio of the rolling moments produced due to the aileron and the differential elevator,

$$\begin{aligned}\xi_d &= \left[\frac{|L_\xi|}{|L_\xi| + |L_{\bar{\eta}}|}\right]\left((\dot{p}_d - \dot{p}_s)\frac{I_{xx}}{L_\xi} + \xi_s\right)\\ \bar{\eta}_d &= \left[\frac{|L_{\bar{\eta}}|}{|L_\xi| + |L_{\bar{\eta}}|}\right]\left((\dot{p}_d - \dot{p}_s)\frac{I_{xx}}{L_{\bar{\eta}}} + \bar{\eta}_s\right)\end{aligned} \tag{12}$$

The steady-state bifurcation results for the HHIRM augmented with an NDI controller for both the longitudinal and lateral planes are shown in Fig. 6. The corresponding control surface deflections are shown in Fig. 7. Again the varying parameter is longitudinal stick input, but the elevator is plotted on the x axis for ease of comparison with Figs. 2 and 4.

The results in Fig. 6 show how the stable region has been extended up to $\alpha = 42$ deg. The lateral states remain at zero up to this point, as

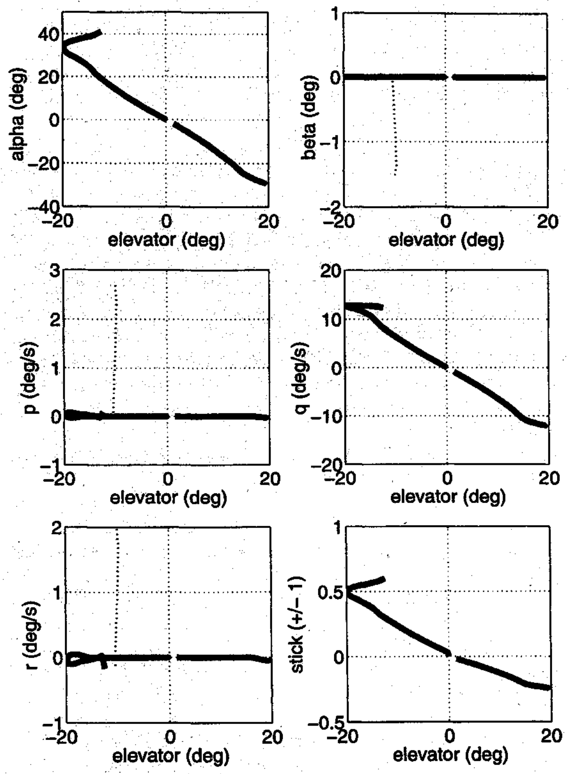

Fig. 6 Steady states for augmented aircraft model (longitudinal and lateral controller): ▬, stable, and - - - -, unstable.

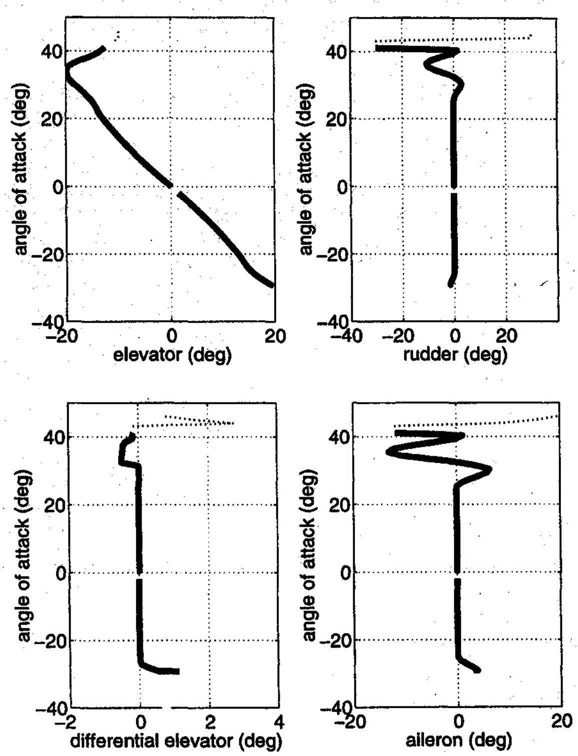

Fig. 7 Control surface deflections (longitudinal and lateral controller): ▬, stable, and - - - -, unstable.

demanded by the inputs to the lateral controller. Above $\alpha = 42$ deg, the aircraft model becomes unstable, and small amounts of lateral dynamics are beginning to couple into the longitudinal motion. In comparison to Fig. 2, it can be seen that the stable branch for $-26 \leq \alpha \leq 26$ deg has joined the previously unstable branch beginning at $\alpha = 38$ deg, so that the entire spinning region has been avoided. Also, the previously unstable region for negative angles of attack has been stabilized.

These results can also be interpreted from the control deflections in Fig. 7. In the range $-26 \leq \alpha \leq 26$ deg, there are no lateral control deflections because there is no lateral coupling. Above $\alpha \approx 26$ deg, there is aerodynamic coupling in the aircraft model, and the rudder and aileron are deflected by the NDI controller to remove these effects. At $\alpha \approx 32$ deg, the differential elevator begins to be also used for additional roll control. At $\alpha = 42$ deg, the rudder saturates in both the maximum and minimum directions, there is no more control power available to remove the coupling, and the aircraft model goes unstable. No further pitching up is available because the differential elevator deflection for roll control leads the symmetric elevator to saturate.

The control law, therefore, has achieved the design specifications of removing the instabilities at which the aircraft is prone to depart into a spin. The stable, decoupled flight region is now $-30 \leq \alpha \leq 42$ deg, an extension of 16 deg more incidence achievable in controlled flight. The results also give an indication of where to impose angle-of-attack limiting, so that the aircraft can operate in the stable region, without going unstable. An angle-of-attack limit of $\alpha = 40$ deg was imposed, and the steady states of the augmented model were again calculated. The results in Fig. 8 clearly show that the aircraft model remains stable for the full incidence range $-30 \leq \alpha \leq 40$ deg. The plot for elevator deflection against longitudinal stick variation shows how the aircraft model remains stable at a constant steady-state condition for increased longitudinal stick deflection beyond that which results in the maximum incidence being achieved.

The results also demonstrate how NDI works in achieving the commanded input by effecting the lateral controls to remove unwanted coupling. This also highlights an important result on the stability of NDI. This analysis has shown that, up to the limits of control surface effectiveness, NDI provides global stability to the system being controlled. Note that global stability is defined here in the sense that the closed-loop aircraft model is stable for slowly varying control demand, not in the sense of robustness to external disturbances or an abrupt change in control demand that may cause the aircraft to depart to regions not connected to the main trim branch. Again, time simulation is not presented due to the adequacy of the results in demonstrating the use of bifurcation analysis as an aid for control law design.

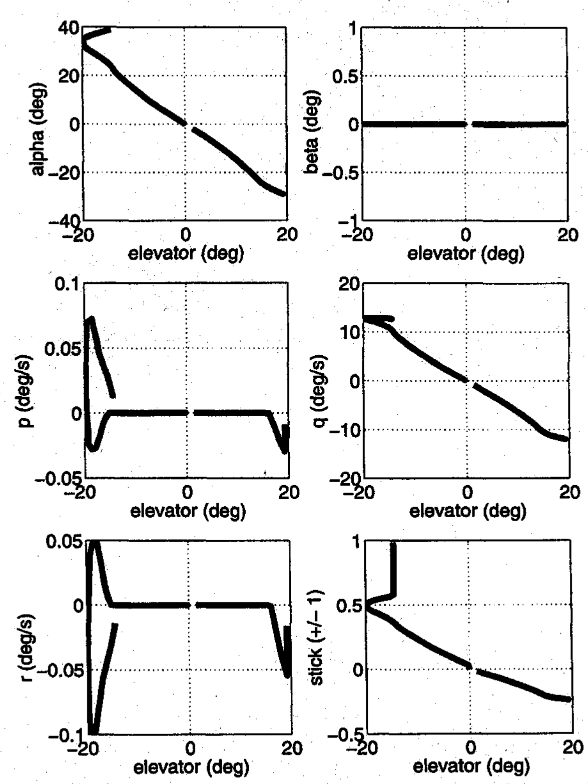

Fig. 8 Steady states of augmented model with incidence limit (longitudinal and lateral controller): ▬, stable, and - - - -, unstable.

Conclusions

This work demonstrates the use of bifurcation analysis techniques as an aid to iterative control law design and subsequent validation. As has been extensively researched, bifurcation analysis is a powerful method for investigating the characteristics of complex aircraft models, both with and without control systems. Here, the analysis results have been used to specify control law requirements, such as stabilization and the removal of departures into spin regions. The use of a nonlinear control law design method has retained the nonlinearity of the entire process, without using linear techniques that may be valid for only a small region around the design point. The analysis of the augmented aircraft model demonstrates the use of bifurcation analysis for validating the control law and for indicating the necessity for any modification of the design. The extension of the stable controlled flight region by 16-deg incidence from $\alpha = 26$ to $\alpha = 42$ deg demonstrates the global stabilization properties of the NDI control law design method.

Acknowledgments

This work was supported by the United Kingdom Ministry of Defence. The authors would like to express their thanks to the first reviewer whose invaluable comments helped to focus this paper.

References

[1]Mehra, R. D., Kessel, W. C., and Carroll, J. V., "Global Stability and Control Analysis of Aircraft at High Angles of Attack," Scientific Systems, Annual Technical Rept. 1, ONR-CR215-248-1, Cambridge, MA, 1977.

[2]Caroll, J. V., and Mehra, R. K., "Bifurcation Analysis of Nonlinear Aircraft Dynamics," *Journal of Guidance*, Vol. 5, No. 5, 1982, pp. 529–536.

[3]Lowenberg, M. H., "Stability and Controllability Evaluation of Sustained Flight Maneuvers," AIAA Paper 96-3422, July 1996.

[4]Lowenberg, M. H., "Bifurcation Analysis as a Tool for Postdeparture Stability Enhancement," AIAA Paper 97-3716, Aug. 1997.

[5]Jahnke, C. C., and Culick, F. E. C., "Application of Bifurcation Theory to High-Angle-of-Attack Dynamics of the F-14," *Journal of Aircraft*, Vol. 31, No. 1, 1994, pp. 26–34.

[6]Jahnke, C. C., "Application of Dynamical Systems Theory to Non-linear Aircraft Dynamics," Ph.D. Dissertation, Jet Propulsion and Mechanical Engineering Dept., California Inst. of Technology, Pasadena, CA, 1990.

[7]Planeaux, J. B., Beck, J. A., and Baumann, D. D., "Bifurcation Analysis of a Model Fighter Aircraft with Control Augmentation," AIAA Paper 90-2836, Aug. 1990.

[8]Planeaux, J. B., and McDonnell, R. J., "Thrust Contributions to the Spin Characteristics of a Model Fighter Aircraft," AIAA Paper 91-2887, Aug. 1991.

[9]Liebst, B. S., and DeWitt, B. R., "Wing Rock Suppression in the F-15 Aircraft," AIAA Paper 97-3719, Aug. 1997.

[10]Avanzini, G., and de Matteis, G., "Bifurcation Analysis of a Highly Augmented Aircraft Model," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 4, 1997, pp. 754–759.

[11]Wang, Z., He, Z., and Lan, C. E., "Nonlinear Analysis of Stability and Control for an F-16 Configuration," AIAA Paper 97-3722, Aug. 1997.

[12]Patel, Y., and Smith, P. R., "Departure Analysis of a Twin-Finned, High Performance Combat Aircraft," AIAA Paper 96-3669, July 1996.

[13]Guicheteau, P., "Bifurcation Theory in Flight Dynamics: An Application to a Real Combat Aircraft," International Council of the Aeronautical Sciences, ICAS Paper 116 (90-5.10.4), Sept. 1990, pp. 1990–1998.

[14]Ananthkrishnan, N., and Sudhakar, K., "Prevention of Jump in Inertia-Coupled Roll Maneuvers of Aircraft," *Journal of Aircraft*, Vol. 31, No. 4, 1993, pp. 981–983.

[15]Jahnke, C. C., and Culick, F. E. C., "Application of Dynamical Systems Theory to Nonlinear Aircraft Dynamics," AIAA Paper 88-4372, Aug. 1988.

[16]Zagaynov, G. I., and Goman, M. G., "Bifurcation Analysis of Critical Aircraft Flight Regimes," International Council of the Aeronautical Sciences, ICAS Paper 84-4.2.1, Sept. 1984.

[17]Enns, D. F., Bugajski, D. J., Hendrick, R. C., and Stein, G., "Dynamic Inversion: An Evolving Methodology for Flight Control," *International Journal of Control*, Vol. 59, No. 1, 1994, pp. 71–91.

[18]Bugajski, D. J., and Enns, D. F., "Nonlinear Control Law with Application to High-Angle-of-Attack Flight," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 3, 1992, pp. 761–767.

[19]Snell, A. S., Enns, D. F., and Garrad, W. L., "Nonlinear Inversion Flight Control for a Supermaneuverable Aircraft," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 4, 1992, pp. 976–984.

[20]Enns, D. F., "Robustness of Dynamic Inversion vs μ-Synthesis: Lateral-Directional Flight Control Example," AIAA Paper 90-3338, Aug. 1990.

[21]Snell, A., "Cancellation Control Law for Lateral-Directional Dynamics of a Supermaneuverable Aircraft," AIAA Paper 93-3775, Aug. 1993.

[22]Smith, P. R., "Functional Control Law Design Using Exact Nonlinear Dynamic Inversion," AIAA Paper 94-3516, Aug. 1994.

[23]Banks, R. E., "PLTMG User's Guide—Edition 5.0," Univ. of California, La Jolla, CA, July 1988.

[24]Patel, Y., and Littleboy, D. M., "Piloted Simulation Tools for Aircraft Departure Analysis," *Philosophical Transactions of the Royal Society of London: Mathematical, Physical, and Engineering Sciences*, Theme Issue on Nonlinear Flight Dynamics for High Performance Aircraft (to be published).